

On MPS construction of blocking sets in projective spaces: a generalization

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Abstract

In this paper we provide a generalization of the MPS construction of blocking sets of $PG(r, q^n)$ using subspaces of dimension $s \leq n - 2$. By this construction, we determine a new non-planar example in $PG(3, q^6)$.

1 Introduction

The notion of *blocking set* was introduced for the first time by Richardson in a game theory setting, see [16], recalling the work of Shapley [17] and of Morgenstern and von Neumann [14] as a *blocking coalition*, that is a set of players which prevents every coalition from winning but it is not itself a winning coalition. Richardson also pointed out an interesting relationship between the theory of blocking sets and projective geometry, as made clear by the following example.

Game 1. Consider $PG(2, p^n)$ as a game where the point set X of $PG(2, p^n)$ is the set of players and, denote by \mathfrak{F} the set of lines of $PG(2, p^n)$. The minimal winning coalitions are the elements of \mathfrak{F} . In this scenario, the blocking sets turn out to be exactly those subsets of X which intersect all of the lines, without containing any.

More generally now, in a projective or affine space, it is usually defined as *k-blocking set* a point set meeting every $(n - k)$ -dimensional subspace, and as *blocking set* a point set meeting every hyperplanes. More generally, in a projective or affine space, it a *blocking set with respect to k-dimensional subspaces* is a point set meeting every k -dimensional subspace at least once. A k -blocking set is *trivial* if it contains a $(n - k)$ -dimensional subspace. In general, any set containing a blocking set is still a blocking set. Thus, we are interested in *minimal* ones with respect to inclusion.

There is a vast literature on blocking sets (more than 400 papers on **math-scinet**). Here, we refer the reader just to [4, 5, 9, 12, 13, 14] which are closely related in techniques and topics to the present work. In [10, 11], Mazzocca, Polverino and Storme have introduced several constructions of blocking sets of $\Pi_r = PG(r, q^n)$ making use of the so called *Barlotti-Cofman representation of $PG(r, q^n)$ in $PG(nr, q)$* . One of these constructions, which we shall denote as *MPS construction*, is recalled in Section 2. In Section 3, a generalization of this construction is introduced and an example of minimal blocking set, which appears not to be equivalent to an MPS one, is obtained and studied in Section 4.

2 The MPS construction

By MPS construction we mean the construction carried out by Mazzocca, Polverino, Storme in [10]: starting from a blocking set in a projective space, one can construct blocking sets in spaces whose order is a power of the original one. The idea of the construction generalizes the planar version of Mazzocca and Polverino in [11]; in this section we follow [6].

We consider the *Barlotti-Coffman* representation of $PG(r, q_1^n)$ in $PG(nr, q_1)$. Take $\Sigma = PG(nr - 1, q_1)$ and let \mathcal{S} be one of its desarguesian $(n - 1)$ -spreads; see [1, 8] and also [3].

Definition 1. *Embed Σ as an hyperplane in $\Sigma' = PG(nr, q_1)$ and define a **point-line geometry** $\Pi_r = \Pi_r(\Sigma', \Sigma, \mathcal{S})$ in the following way:*

- *the points of Π_r are the points of $\Sigma' \setminus \Sigma$ and the elements of \mathcal{S} ;*
- *the lines of Π_r are the n -subspaces of Σ' intersecting Σ in an element of \mathcal{S} , and the lines of $PG(\mathcal{S})$;*
- *the point-line incidences are inherited from Σ and Σ' .*

Theorem 1 ([2]). *The incidence structure Π_r is isomorphic to the projective space $PG(r, q)$, where $q = q_1^n$.*

This incident structure is called the **Barlotti-Cofman representation of $PG(r, q)$** .

The points of Π_r in $\Sigma' \setminus \Sigma$ are called *affine*. Let Y be a fixed element of \mathcal{S} and let $\Omega = \Omega_{n-2}$ be an hyperplane of Y . Let $\Gamma' = \Gamma'_{(r-1)n+1}$ be an $((r - 1)n + 1)$ -subspace of Σ' disjoint from Ω . Also, denote by $\Gamma = \Gamma_{(r-1)n}$ the $(r - 1)n$ -subspace intersection of Γ' and Σ and by t the intersection point of Γ and Y .

Let \overline{B} be a blocking set of Γ' such that $\overline{B} \cap \Gamma = \{\tilde{q}\}$, \tilde{q} a point, with the following property:

$$L \setminus \{t\} \not\subset \overline{B},$$

for every line L of Γ' through t .

Denote by $K = \mathfrak{K}(\Omega, \overline{B})$ the cone with vertex Ω and base \overline{B} . Note that, since $\Gamma' \cap \Omega = \emptyset$, we have $\langle \overline{p}, \Omega \rangle \cap \langle \overline{p'}, \Omega \rangle = \Omega$, for any two distinct points $\overline{p'}, \overline{p} \in \overline{B}$. Let B be the subset of Π_r defined by

$$B = (K \setminus \Sigma) \cup \{X \in \mathcal{S} : X \cap K \neq \emptyset\},$$

and note that

- if $\tilde{q} \in Y$ (i.e. $t = \tilde{q}$), then $|B| = q_1^{n-1}(|\overline{B}| - 1) + 1$ and $B \cap PG(\mathcal{S}) = \{Y\}$;
- while if $\tilde{q} \notin Y$, then $|B| = q_1^{n-1}|\overline{B}| + 1$ and $|B \cap PG(\mathcal{S})| = q_1^{n-1} + 1$.

In both cases we get:

Theorem 2 ([10]). *B is a blocking set of the projective space Π_r .*

Suppose that \overline{B} is a minimal blocking set of Γ' such that $t = \tilde{q}$ (this case is called MPS construction A), in other words, suppose that Γ is a tangent hyperplane of \overline{B} at the point \tilde{q} .

In this case, $B = (K \setminus \Sigma) \cup \{Y\}$ and

$$|B| = |K \setminus \{\Sigma\}| + 1 = q_1^{n-1}(|\overline{B}| - 1) + 1.$$

Then the following theorem holds true; see [10].

Theorem 3. *B is a minimal blocking set of Π_r if and only if \overline{B} is a minimal blocking set of Γ' .*

3 The construction

Let us consider $\Pi_r = PG(r, q^n)$ as represented (using the Barlotti-Cofman representation) into $PG(nr, q)$ with respect to a desarguesian spread \mathcal{S} of an hyperplane Σ of $PG(nr, q)$.

Let X be a fixed element of \mathcal{S} and Ω be a subspace of X of dimension $s \leq n - 2$. Let Γ be a subspace of Σ such that $\dim \Gamma = rn - s - 2$ and $\Gamma \cap \Omega = \emptyset$, let $\Theta = \Gamma \cap X$ and Γ' be a subspace of $PG(nr, q)$ of dimension $rn - s - 1$ such that $\Gamma' \cap \Sigma = \Gamma$. Let $\mathfrak{F} = \mathfrak{F}_{(r-1)n}^s = \{\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma' : S_{(r-1)n} \text{ hyperplane of } \Pi_r \text{ not containing } X\}$. Because $\Gamma' \cap \Omega = \emptyset$ by the Grassmann formula we get:

$$\dim(\langle \Gamma', \Omega \rangle) - 1 = rn - s - 1 + s$$

hence

$$\dim(\langle \Gamma', \Omega \rangle) = rn = \dim(\langle \langle \Gamma', \Omega \rangle, S_{(r-1)n} \rangle)$$

and, called $I_{(r-1)n}$ an element of \mathfrak{F} , we get:

$$\begin{aligned} \dim(\langle \langle \Omega, S_{(r-1)n} \rangle, \Gamma' \rangle) + \dim(\langle \langle \Omega, S_{(r-1)n} \rangle \cap \Gamma' \rangle) &= rn + \dim(I_{(r-1)n}) = \\ &= \dim(\langle \langle \Omega, S_{(r-1)n} \rangle \rangle) + \dim(\Gamma') = (r-1)n + s + 1 + rn - s - 1 = 2rn - n. \end{aligned}$$

It follows that $\dim(I_{(r-1)n}) = (r-1)n$ and \mathfrak{F} is a family of subspaces of $PG(nr, q)$ of dimension $(r-1)n$.

Definition 2. *A subset \overline{B} of Γ' is called **blocking set with respect to \mathfrak{F}** or simply an **\mathfrak{F} -blocking set** if given $I_{(r-1)n} \in \mathfrak{F}$, we get $\overline{B} \cap I_{(r-1)n} \neq \emptyset$. An **\mathfrak{F} -blocking set** is called **minimal** if it is minimal with respect to the inclusion.*

*Moreover, in the case \mathfrak{F} be a family of subspaces of $PG(m, q)$ of dimension k and B an \mathfrak{F} -blocking set, we call B **trivial** if it contains a subspace of dimension $m - k$.*

Let \overline{B} be an \mathfrak{F} -blocking set of Γ' such that $\overline{B} \cap \Sigma = \Theta$, then we consider the cone $K = \mathfrak{K}(\Omega, \overline{B})$ and $B = K \cup \{X\}$. We call this construction “**generalized MPS construction**”.

Theorem 4. *$B \cap S_{(r-1)n} \neq \emptyset$ for all hyperplanes $S_{(r-1)n}$ of Π_r .*

Proof. Consider an hyperplane $S_{(r-1)n}$ not containing X . Hence there exists $p' \in \langle S_{(r-1)n}, \Omega \rangle \cap \overline{B}$ and $p \in \mathfrak{K}(\Omega, \overline{B}) \cap S_{(r-1)n}$. \square

Now we introduce some useful lemmas for proving the minimality of B .

Lemma 1. *Let P_1, P_2 be subspaces of a projective space Γ such that $P_1 \cap P_2 = \emptyset$. Then $\langle \overline{p}, P_1 \rangle \cap \langle \overline{p'}, P_1 \rangle = P_1$ for any two distinct points $\overline{p}, \overline{p'} \in P_2$.*

Proof. Suppose there is an intersection point outside P_1 , this means that $\langle \overline{p}, P_1 \rangle = \langle \overline{p'}, P_1 \rangle$. Then the line through \overline{p} and $\overline{p'}$ meets P_1 , which implies that P_2 meets P_1 non-trivially, a contradiction. \square

Lemma 2. *Let us consider $\overline{B} \setminus X$, then for $I_{(r-1)n} = \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma' \in \mathfrak{F}$ and $S_{(r-1)n}$ not containing X we get:*

$$|B \cap S_{(r-1)n}| = |(\overline{B} \setminus X) \cap I_{(r-1)n}|.$$

Proof. Consider:

$$B \cap S_{(r-1)n} = (K \setminus \Sigma) \cap S_{(r-1)n} = \bigcup_{\overline{p} \in \overline{B} \setminus X} (\langle \overline{p}, \Omega \rangle \cap S_{(r-1)n})$$

Since $\dim(\langle \Omega, S_{(r-1)n} \rangle) = (s+1) + (r-1)n$ and, for $\overline{p} \in \overline{B} \setminus X$

$$\dim(\langle \overline{p}, \Omega \rangle \cap S_{(r-1)n}) + \dim(\langle \overline{p}, \Omega, S_{(r-1)n} \rangle) = (s+1) + (r-1)n,$$

we get $\dim(\langle \overline{p}, \Omega \rangle \cap S_{(r-1)n}) \in \{-1, 0\}$ and equals 0 if and only if $\overline{p} \in \langle \Omega, S_{(r-1)n} \rangle$ and since $\overline{p} \in \overline{B} \setminus X \subseteq \Gamma' \setminus \Sigma$ this holds if and only if $\overline{p} \in (\overline{B} \setminus X) \cap I_{(r-1)n}$. Since (by Lemma 1) $\langle \overline{p}, \Omega \rangle \cap \langle \overline{p'}, \Omega \rangle = \Omega$ for any two distinct points $\overline{p}, \overline{p'} \in \overline{B}$ we get:

$$\bigcup_{\overline{p} \in \overline{B} \setminus X} (\langle \overline{p}, \Omega \rangle \cap S_{(r-1)n}) = \bigsqcup_{\overline{p} \in (\overline{B} \setminus X) \cap I_{(r-1)n}} (\langle \overline{p}, \Omega \rangle \cap S_{(r-1)n}).$$

The claim follows by a simple counting. \square

Theorem 5. *Let $\overline{B} \setminus X$ be a minimal (and such that \overline{B} is non-trivial) \mathfrak{F} -blocking set, then B is a minimal (non-trivial) blocking set of Π_r .*

Proof. Let $\overline{B} \setminus X$ be a minimal blocking set, then by Theorem 4, $(B \setminus \{X\}) \cap S'_{(r-1)n} \neq \emptyset$ for all hyperplanes $S'_{(r-1)n}$ not containing $\{X\}$.

Consider now a point $p \in B \setminus \{X\}$ then there exists at least one point \tilde{q} such that:

$$\tilde{q} \in \overline{B} \cap \langle p, \Omega \rangle.$$

Let now $\tilde{q}, \tilde{q'}$ be such that $\tilde{q}, \tilde{q'} \in \overline{B} \cap \langle p, \Omega \rangle$ which implies $\langle p, \Omega \rangle = \langle \tilde{q}, \Omega \rangle = \langle \tilde{q'}, \Omega \rangle$. Since, for all $\tilde{q} \neq \tilde{q'} \in \overline{B}$ we have $\langle \tilde{q}, \Omega \rangle \cap \langle \tilde{q'}, \Omega \rangle = \Omega$ we get $\tilde{q} = \tilde{q'}$. Hence the point \tilde{q} is uniquely determined by the point p .

Because of the minimality of $\overline{B} \setminus X$ there exists $I_{(r-1)n} = \langle \Omega, \overline{S}_{(r-1)n} \rangle \cap \Gamma'$ such that $\{\tilde{q}\} = I_{(r-1)n} \cap \overline{B} \setminus X$. We consider now the hyperplane of Π_r represented by:

$$S_{(r-1)n} = \langle p, \overline{S}_{(r-1)n} \cap \Sigma \rangle.$$

Then, because $\tilde{q} \in \langle p, \Omega \rangle$ and $\tilde{q} \in \langle \Omega, S_{(r-1)n} \rangle$ we have:

$$\begin{aligned} \langle \Omega, S_{(r-1)n} \rangle \cap \Gamma' &= \langle \Omega, p, \overline{S}_{(r-1)n} \cap \Sigma \rangle \cap \Gamma' = \\ &= \langle \Omega, \tilde{q}, \overline{S}_{(r-1)n} \cap \Sigma \rangle \cap \Gamma' = \langle \Omega, \overline{S}_{(r-1)n} \rangle \cap \Gamma' = I_{(r-1)n}. \end{aligned}$$

Hence, since $|I_{(r-1)n} \cap (\overline{B} \setminus X)| = |\{\tilde{q}\}| = 1$, because of Lemma 2, $S_{(r-1)n}$ intersects B exactly in the point p .

Suppose now B contains a line of Π_r , i.e. $\exists S_n : S_n \subseteq B$. Hence $S_n = B$ by minimality and $S_n = \bigcup_{\overline{p} \in \overline{B}} \langle \overline{p}, \Omega \rangle$. Since $\Gamma' \cap \Omega = \emptyset$ for $\overline{p} \in \overline{B} \subseteq \Gamma'$ we have $\Gamma' \cap \langle \overline{p}, \Omega \rangle = \overline{p}$ and hence:

$$\Gamma' \cap S_n = \Gamma' \cap (\bigcup_{\overline{p} \in \overline{B}} \langle \overline{p}, \Omega \rangle) = \bigcup_{\overline{p} \in \overline{B}} (\Gamma' \cap \langle \overline{p}, \Omega \rangle) = \bigcup_{\overline{p} \in \overline{B}} \overline{p} = \overline{B}.$$

Therefore if B is trivial then \overline{B} is trivial. \square

Theorem 6. *In the situation of the previous theorem the following equality holds:*

$$|B| = (|\overline{B}| - \frac{q^{n-s-1} - 1}{q - 1})q^{s+1} + 1$$

Proof. For each $\overline{p}, \overline{p}' \in \overline{B} \setminus \{X\}$ we have:

$$\langle \overline{p}, \Omega \rangle \cap \langle \overline{p}', \Omega \rangle = \Omega;$$

because $B \cap PG(\mathcal{S}) = \{X\}$ we get:

$$|B| = (|\overline{B}| - |\overline{B} \cap \Sigma|)(|\langle \overline{p}, \Omega \rangle| - |\Omega|) + 1$$

and because $\overline{B} \cap \Sigma = \Theta$ the claim follows easily. \square

Remark 1. *If $r = 2$ the above construction coincides with the second construction given by Mazzocca and Polverino in [11] (MP Construction B); if $s = n-2$ the above construction coincides with the first construction given by Mazzocca, Polverino and Storme in [10] (MPS Construction A).*

Our goal is now to find some example of minimal \mathfrak{F} -blocking sets $\overline{B} \setminus X$ in order to use our construction and obtain minimal blocking sets with respect to hyperplanes.

4 Non-planar example

Let consider $\Pi_3 = PG(3, q^6)$ as represented (via Barlotti-Cofman representation) with respect to the desarguesian 2-spread \mathcal{S} of an hyperplane Σ of $PG(9, q^2)$. Let X, X' be two fixed elements of \mathcal{S} and let p be a point of X . Let Γ be a subspace of Σ of dimension 7 such that $X' \subseteq \Gamma$ and $p \notin \Gamma$. Let $\Theta = \Gamma \cap X = \langle r, \tilde{q} \rangle$ for $r, \tilde{q} \in \Theta$ and let Γ' be a subspace of $PG(9, q^2)$ of dimension 8 such that $\Gamma' \cap \Sigma = \Gamma$. Let $\mathfrak{F} := \{\langle S_6, p \rangle \cap \Gamma' : S_6 \text{ is an hyperplane of } \Pi_3 \text{ not containing } X\}$. Moreover let us consider the family of seven dimensional subspaces of $PG(9, q^2)$ defined by:

$$\mathfrak{H} := \{\langle S_6, p \rangle : S_6 \text{ is an hyperplane of } \Pi_3 \text{ not containing } X\}.$$

It is clear that a subset of Γ' is a minimal \mathfrak{F} -blocking set if and only if it is a minimal \mathfrak{H} -blocking set.

Let π be a plane of Γ' through the point \tilde{q} and a point $t \in X'$, not contained in Σ , that is $\pi \cap \Sigma = \langle t, \tilde{q} \rangle$. Let V be a Baer subplane of π such that $V \cap \langle t, \tilde{q} \rangle = \{\tilde{q}\}$.

We recall that every line L of π intersects V in either 1 or $q + 1$ points: in the first case we say that L is an **imaginary** line for V and in the second case we say that L is a **real** line for V . It is possible to prove that for each point u of $\pi \setminus V$ there exists a unique real line $L : u \in L$.

Name by S_3 the 3-space containing π and r , we consider the Baer cone $\mathfrak{K}(r, V)$ and we observe that every line L of S_3 not through the point r , is either a real line or an imaginary line for some Baer subplane contained in the cone $\mathfrak{K}(r, V)$ and so intersects the cone in either 1 or $q + 1$ points: in the first case we say that L is an **imaginary** line for the cone $\mathfrak{K}(r, V)$ and in the second case we say that L is a **real** line for $\mathfrak{K}(r, V)$.

Let us consider a real line L of the plane π through the point $t \in X'$, a point $s \in V \cap L$ and we construct the cone:

$$\bar{B} := (\mathfrak{K}(r, V) \setminus \mathfrak{K}(r, L)) \cup \mathfrak{K}(r, s) = \mathfrak{K}(r, V \setminus L \cup \{s\}).$$

Then we have the following Proposition.

Proposition 1. *The cone \bar{B} is a blocking set with respect to the family of subspaces*

$$\mathfrak{H}_t := \{S_7 \in \mathfrak{H} : X' \subseteq S_7\}.$$

Proof. Since the cone $\mathfrak{K}(r, V)$ is a Baer Cone of type $(2, 0)$, it is a blocking set with respect to the seven dimensional subspaces of $PG(9, q^2)$ (see [6],[7]) and hence a \mathfrak{H} -blocking set. Therefore the elements of \mathfrak{H}_t are in the form $\langle p, u, X', Y_1 \rangle$ with $Y_1 \in \mathcal{S}$, $Y_1 \not\subset \langle X', X \rangle$ and $u \in \mathfrak{K}(r, V)$. Then, if $u \in \bar{B}$, we get that $\langle p, u, X', Y_1 \rangle \cap \bar{B} \neq \emptyset$. Let us suppose $u \notin \bar{B}$, hence $u \in \mathfrak{K}(r, L \setminus \{s\})$. But in this case $\langle t, u \rangle \cap \mathfrak{K}(r, s) \neq \emptyset$ and hence, since $t \in X'$:

$$\langle p, u, X', Y_1 \rangle \cap \bar{B} \supseteq \langle t, u \rangle \cap \bar{B} \supseteq \langle t, u \rangle \cap \mathfrak{K}(r, s) \neq \emptyset.$$

We conclude that \bar{B} is an \mathfrak{H}_t -blocking set. \square

Now we construct a set \tilde{B} such that $\bar{B} \cup \tilde{B}$ is a non-planar, minimal \mathfrak{H} -blocking set. For doing this we need the following lemma.

Lemma 3. *There exists exactly one point $\tilde{t} \in X'$, $\tilde{t} \neq t$ such that for all $S_2 \in \mathcal{S}$, $S_2 \subset \langle X, X' \rangle$ with $\tilde{t} \in \langle p, S_2 \rangle$ we have $\langle p, S_2 \rangle \cap \langle t, r \rangle \neq \emptyset$.*

Proof. Since the spread \mathcal{S} is desarguesian the set $\mathfrak{R} := \{S_2 \in \mathcal{S} : S_2 \cap \langle t, r \rangle \neq \emptyset\}$ is a regulus. Clearly $X \in \mathfrak{R}$ and hence there exists exactly one line l such that $p \in l$ and $\mathfrak{R} = \{S_2 \in \mathcal{S} : S_2 \cap l \neq \emptyset\}$. Since also $X' \in \mathfrak{R}$ we have $X' \cap l \neq \emptyset$; let we call $\tilde{t} = X' \cap l$. Then, $l = \langle p, \tilde{t} \rangle$ and hence:

$$\mathfrak{R} = \{S_2 \in \mathcal{S} : S_2 \cap l \neq \emptyset\} = \{S_2 \in \mathcal{S} : \tilde{t} \in \langle p, S_2 \rangle\}.$$

But, for the definition of \mathfrak{R} this means that for all $S_2 \in \mathcal{S} : \tilde{t} \in \langle p, S_2 \rangle$ we have:

$$\emptyset \neq S_2 \cap \langle t, r \rangle \subseteq \langle p, S_2 \rangle \cap \langle t, r \rangle.$$

Let now consider $S_2 \in \mathcal{S}$ such that $\langle p, S_2 \rangle \cap \langle t, r \rangle \neq \emptyset$ and let $u \in \langle p, S_2 \rangle \cap \langle t, r \rangle$. Let us consider $S'_2 \in \mathcal{S}$ such that $u \in S'_2$; then, since $\langle p, u \rangle \subseteq \langle p, S_2 \rangle \cap \langle p, S'_2 \rangle$ we must have $S_2 = S'_2$. Therefore $u \in S_2 \cap \langle t, r \rangle$ and hence $S_2 \in \mathfrak{R}$ and

$$\mathfrak{R} = \{S_2 \in \mathcal{S} : \langle p, S_2 \rangle \cap \langle t, r \rangle \neq \emptyset\}.$$

Since for a point $y \in X'$ different from \tilde{t} the regulus

$$\mathfrak{R}' := \{S_2 \in \mathcal{S} : y \in \langle p, S_2 \rangle\}$$

is different from \mathfrak{R} , and hence contains an element not contained in \mathfrak{R} , we have that there exists a unique required point $\tilde{t} \in X'$. \square

Let us consider lines $L_1, L_2, L_3 \subseteq X'$ such that $L_1 \cap L_2 \cap L_3 = \emptyset$ and $t, \tilde{t} \notin L_i$ $i \in \{1, 2, 3\}$ where \tilde{t} is the point of the previous Lemma. Let us consider a point $h \in \Gamma'$, $h \notin \langle X, X', V \rangle$, then we define:

$$\tilde{B} := \mathfrak{R}(h, L_1 \cup L_2 \cup L_3) \setminus \Sigma.$$

Then:

Proposition 2. *\tilde{B} is a blocking set with respect to the family of subspaces*

$$\mathfrak{H} \setminus \mathfrak{H}_t := \mathfrak{H} \setminus \{S_7 \in \mathfrak{H} : X' \subseteq S_7\}.$$

Proof. Given $S_7 \in \mathfrak{H} \setminus \mathfrak{H}_t$, we have:

$$S_7 \cap \mathfrak{R}(h, X') \supseteq R \equiv PG(1, q^2) : R \not\subseteq X'.$$

Since $\mathfrak{R}(h, X') \setminus X' \equiv A(3, q^2)$ and the union of three planes, two by two non-parallel, is a blocking set with respect to the lines in $A(3, q^2)$ we have that \tilde{B} is a blocking set with respect to $\mathfrak{H} \setminus \mathfrak{H}_t$. \square

4.1 Minimality

Now we characterize some property of the spectrum of intersection between $\bar{B} \cup \tilde{B}$ and the elements of \mathcal{S} . Let we call $S_3 := \langle \bar{B} \rangle$ and $S_2 := S_3 \cap \Sigma$.

Proposition 3. *Let $S_7 \in \mathfrak{H}$, then $S_7 \cap S_3$ is a line not contained in Σ and hence $|S_7 \cap \bar{B}| \in \{0, 1, q, q+1\}$.*

In particular if $S_7 \cap S_3$ is a real line of S_3 (w.r.t. the Baer Cone $\mathfrak{R}(r, V)$) through the point t , then it is contained in the plane $\langle t, r, s \rangle$.

Proof. Let $S_7 \in \mathfrak{H}$, then $S_7 = \langle p, Z, Z', n \rangle$ for some $n \notin \Sigma$ and for some $Z, Z' \in \mathcal{S}$ with $X \not\subseteq \langle Z, Z' \rangle$.

For the Grassmann formula we have that $\dim(S_3 \cap S_7) \geq 1$. Let us suppose that $\dim(S_3 \cap S_7) \geq 2$ which means that there exists a plane $\pi_2 \subseteq S_3 \cap S_7$ and therefore a line $L \subseteq (S_3 \cap S_7) \cap \Sigma$; since $p \notin S_3$ we get $p \notin L$.

We have that $L \subseteq S_3 \cap \Sigma = \langle r, \tilde{q}, t \rangle$ and hence that $L \cap \langle r, \tilde{q} \rangle \subseteq L \cap X \neq \emptyset$. But since

$$\{p\} = S_7 \cap X \supseteq L \cap X \neq \emptyset$$

and $p \notin L$ we get a contradiction. Therefore $\dim(S_3 \cap S_7) = 1$ and, similarly, we have $\dim((S_3 \cap S_7) \cap \Sigma) = 0$.

Let $L_{S_7} = S_7 \cap S_3$ and note that $r \notin L_{S_7}$ since $X \not\subseteq S_7$. By definition, if the line L_{S_7} is imaginary then $|S_7 \cap \mathfrak{K}(r, V)|$ is 1 and if the line L_{S_7} is real $|S_7 \cap \mathfrak{K}(r, V)| = q + 1$. Then we can get the intersection with \bar{B} : if the line L_{S_7} does not intersect $\mathfrak{K}(r, V) \setminus \mathfrak{K}(r, (V \setminus L) \cup \{s\})$ the intersection is equal to $|S_7 \cap \mathfrak{K}(r, V)|$ otherwise if $L_{S_7} \not\subseteq \langle r, L \rangle$ the intersection decreased by one. Lastly, if $L_{S_7} \subseteq \langle r, L \rangle$ the intersection is always one.

Therefore we have the following possibilities for the intersection $L_{S_7} \cap \bar{B}$:

Line $S_7 \cap \langle \bar{B} \rangle$	$S_7 \cap \bar{B}$
Imaginary	0
Real	1
Imaginary	1
Real	$q + 1$
Real	q

□

Proposition 4. *Let $S_7 \in \mathfrak{H}_t$, then:*

- $S_7 \cap S_3$ is a line through t not contained in Σ .
- $S_7 \cap \tilde{B} = \emptyset$ or $S_7 \cap \tilde{B} = \tilde{B}$.

Proof. Since $X' \subseteq S_7$, $t \in S_7 \cap S_3$, hence by Prop. 3, the intersection $S_7 \cap S_3$ is a line through t not contained in Σ .

Let $\tilde{S}_3 = \langle \tilde{B} \rangle$, i.e. $\tilde{S}_3 = \langle X', h \rangle$. Then either $\tilde{S}_3 \subseteq S_7$ or $\tilde{S}_3 \cap S_7 = X'$. In the first case $S_7 \cap \tilde{B} = \tilde{B}$, in the second case $S_7 \cap \tilde{B} = \emptyset$. □

On the other hand for an element $S_7 \in \mathfrak{H} \setminus \mathfrak{H}_t$ (which means that $X' \not\subseteq S_7$) we have:

Proposition 5. *Let $S_7 \in \mathfrak{H}$ such that $X' \not\subseteq S_7$. Then we have that $S_7 \cap \langle \tilde{B} \rangle$ is a line L'_{S_7} not contained in Σ and hence $|S_7 \cap \tilde{B}| \in \{1, 2, 3, q^2\}$.*

Proof. Let $S_7 = \langle p, u, Z, Z' \rangle$ for some $Z, Z' \in \mathcal{S}$ such that $X, X' \not\subseteq \langle Z, Z' \rangle$ and $u \notin \Sigma$. Since $\langle u, Z, Z' \rangle$ can be seen as a plane of Π_3 and $\langle h, X' \rangle$ as a line not contained in $\langle u, Z, Z' \rangle$, we have that $\langle u, Z, Z' \rangle \cap \langle h, X' \rangle = \langle u, Z, Z' \rangle \cap \langle \tilde{B} \rangle$ is just a point. Therefore $S_7 \cap \langle \tilde{B} \rangle$ is a line L'_{S_7} not contained in Σ . Now, since \tilde{B} is the union of three non parallel affine planes we have $|S_7 \cap \tilde{B}| = |L'_{S_7} \cap \tilde{B}| \in \{1, 2, 3, q^2\}$. □

Hence if $S_7 \in \mathfrak{H}$, by Prop. 4 and 5 we have the following possibilities:

Intersection	$S_7 \cap \tilde{B}$
\emptyset	0
$\langle \tilde{B} \rangle$	$ \tilde{B} $
Line	q^2
Line	3
Line	2
Line	1

Now we apply this result.

Proposition 6. *Let $u \in \bar{B}$, then there exists an element $S_7 \in \mathfrak{H}_t$ tangent to $\bar{B} \cup \tilde{B}$ in u .*

Proof. Let us suppose $u \in \bar{B} \setminus \{s\}$, we determine an element of the family \mathfrak{H} tangent to u . Because in Π_3 exists a plane through $s, h, \{X'\}$, there exists $Y \in \mathcal{S}$ such that $h \in \langle s, Y, X' \rangle$. In particular $Y \notin \langle X, X' \rangle$. Then, called $S_7 = \langle p, u, Y, X' \rangle$ we have $S_7 \in \mathfrak{H}_t$ and hence, for Proposition 4:

$$S_7 \cap S_3 = \langle t, u \rangle.$$

If the line $\langle t, u \rangle$ is imaginary then we have:

$$S_7 \cap \bar{B} = \{u\}.$$

If the line $\langle t, u \rangle$ is real then we have $\langle t, u \rangle \subseteq \langle t, s, r \rangle$ and hence:

$$\begin{aligned} S_7 \cap \bar{B} &= (S_7 \cap S_3) \cap \bar{B} = \\ &= \langle t, u \rangle \cap (\bar{B} \cap \langle t, s, r \rangle) = \langle t, u \rangle \cap \langle r, s \rangle = \{u\}. \end{aligned}$$

For proving that S_7 is tangent we have to estimate the intersection with \tilde{B} . Because of the choice of Y , we have

$$\langle p, h, Y, X' \rangle = \langle p, s, Y, X' \rangle,$$

and, since we have proven that $\bar{B} \cap S_7 = \{u\}$, we have $s \notin S_7$ and:

$$\langle p, h, Y, X' \rangle = \langle p, s, Y, X' \rangle \neq \langle p, u, Y, X' \rangle = S_7$$

and hence $h \notin S_7$. Therefore we have that $S_7 \cap \tilde{B} \neq \tilde{B}$ and, being $S_7 \in \mathfrak{H}_t$, for Proposition 4 we have $S_7 \cap \tilde{B} = \emptyset$. Hence S_7 is tangent to $\bar{B} \cup \tilde{B}$ in u .

Let now $u = s$, and let $s' \in \bar{B} \setminus \{X'\}$, $s' \neq s$. Because, in Π_3 exists a plane through $s', h, \{X'\}$, there exists $Y_1 \in \mathcal{S} \setminus \{X'\}$ such that $h \in \langle s', Y_1, X' \rangle$. Called, as before, $S_7 = \langle p, s, Y_1, X' \rangle$, we have that $S_7 \in \mathfrak{H}_t$ and, for Proposition 4:

$$S_7 \cap \bar{B} = (S_7 \cap S_3) \cap \bar{B} = \langle t, s \rangle \cap \bar{B} = \langle t, s \rangle \cap \bar{B} = \{s\}.$$

For proving that S_7 is tangent we have to estimate again the intersection with \tilde{B} . Because of the choice of Y_1 , we have

$$\langle p, h, Y_1, X' \rangle = \langle p, s', Y, X' \rangle$$

and since we have proven that $\bar{B} \cap S_7 = \{s\}$ and hence $s' \notin S_7$ we have:

$$\langle p, h, Y_1, X' \rangle = \langle p, s', Y_1, X' \rangle \neq \langle p, s, Y_1, X' \rangle = S_7$$

and hence $h \notin S_7$. Therefore we have that $S_7 \cap \tilde{B} \neq \tilde{B}$ and, being $S_7 \in \mathfrak{H}_t$, for Proposition 4 we have $S_7 \cap \tilde{B} = \emptyset$. Hence we can find a tangent element of \mathfrak{H} to all $u \in \bar{B}$ \square

Proposition 7. *Let $u \in \tilde{B}$, then there exists an element $S_7 \in \mathfrak{H} \setminus \mathfrak{H}_t$ tangent to $\bar{B} \cup \tilde{B}$ in u .*

Proof. Let now $u \in \tilde{B}$ and let $L_u = \langle \tilde{p}, u \rangle$ be a line through u such that $\tilde{p} \in X'$, $\tilde{p} \neq \tilde{t}$ and L_u is tangent to \tilde{B} in u .

Because of Lemma 3 there exists an element $Y \in \mathcal{S}$, $Y \subseteq \langle X, X' \rangle$, with $Y \neq X, X'$ such that $\tilde{p} \in \langle p, Y \rangle$ (i.e. $\{\tilde{p}\} = \langle p, Y \rangle \cap X'$) and $\langle p, Y \rangle \cap \langle t, r \rangle = \emptyset$.

Since $\langle p, Y \rangle \cap S_3 = \langle p, Y \rangle \cap \langle r, \tilde{q}, t \rangle$ and $\langle p, Y \rangle \cap \langle r, t \rangle = \emptyset$, the intersection $\langle p, Y \rangle \cap \langle r, \tilde{q}, t \rangle$ is just a point, say m_1 and we have $m_1 \notin \langle t, r \rangle$. Therefore there exists $m_2 \in (L \setminus \{s\}) \cap V$ such that the line $\langle m_1, m_2 \rangle$ is imaginary, i.e. tangent to $\mathfrak{K}(r, V)$ at the point m_2 and hence has empty intersection with \bar{B} . Since in Π_3 there exists a plane through $u, m_2, \{Y\}$, there exists $Z \in \mathcal{S}$ such that $u \in \langle m_2, Z, Y \rangle$.

Let us consider $S_7 := \langle p, m_2, Z, Y \rangle \in \mathfrak{H}$, since $Y \subseteq \langle X, X' \rangle$ and $X \not\subseteq \langle Z, Y \rangle$ we have that $X' \not\subseteq \langle Z, Y \rangle$ and hence $S_7 \notin \mathfrak{H}_t$. Now, we want to prove that S_7 is tangent to $\bar{B} \cup \tilde{B}$ in u . For Proposition 3 we have that $S_7 \cap S_3$ is a line and hence, since $m_1, m_2 \in S_7$ we have: $S_7 \cap S_3 = \langle m_1, m_2 \rangle$. Therefore:

$$S_7 \cap \bar{B} = (S_7 \cap S_3) \cap \bar{B} = \langle m_1, m_2 \rangle \cap \bar{B} = \emptyset.$$

On the other hand, since $S_7 \notin \mathfrak{H}_t$, for Lemma 5, $S_7 \cap \langle \tilde{B} \rangle$ is a line not contained in Σ and hence, since $u, \tilde{p} \in S_7$, we have $S_7 \cap \langle \tilde{B} \rangle = L_u$. Therefore:

$$S_7 \cap \tilde{B} = (S_7 \cap \langle \tilde{B} \rangle) \cap \tilde{B} = L_u \cap \tilde{B} = \{u\}.$$

Hence we can find a tangent element of \mathfrak{H} to all $u \in \tilde{B}$. □

Summing up the previous results we get:

Theorem 7. *$\bar{B} \cup \tilde{B}$ is a minimal and non planar \mathfrak{H} -blocking set. Therefore the cone $B := \mathfrak{K}(p, \bar{B} \cup \tilde{B})$ is a minimal blocking set of $\Pi_3 = PG(3, q^6)$.*

We note that, being non planar it is impossible to obtain this example using the classical MP construction.

Now we see that the blocking set B appears not to be in the MPS class.

Theorem 8. *$B = \mathfrak{K}(p, \bar{B} \cup \tilde{B})$ is a minimal blocking sets of $\Pi_3 = PG(3, q^6)$ not equivalent to any blocking sets of class MPS.*

Proof. First of all, let us evaluate the cardinality of $B = \mathfrak{K}(p, \bar{B} \cup \tilde{B})$.

We have that $\bar{B} = \mathfrak{K}(r, V \setminus L \cup \{s\})$ and hence:

$$|\bar{B} \setminus \Sigma| = q^2(|V \setminus L \cup \{s\}| - |V \cap \Sigma|) = q^2((q^2 + 1) - 1) = q^4.$$

Since \tilde{B} is the union of three non parallel affine planes that share the same point we have that:

$$|\tilde{B}| = 3q^4 - 3q^2 + 1.$$

Therefore we have that:

$$|B| = |\mathfrak{K}(p, \bar{B} \cup \tilde{B})| = q^2(|(\bar{B} \cup \tilde{B}) \setminus \Sigma|) + 1 = q^2(3q^4 - 3q^2 + 1 + q^4) + 1 = 4q^6 - 3q^4 + q^2 + 1.$$

Let $q = p^e$ and let us suppose B is obtained using the MPS construction starting from $PG(3n, q' = p^t)$ and $n \geq 2$, hence $6e = nt$. Then we have seen (cf. [10], pag. 100) that $(p^t)^{n-1} \mid (|B| - 1)$. Since $|B| = q^2(4q^4 - 3q^2 + 1) + 1$ and $p \nmid 4q^4 - 3q^2 + 1$, we have that $p^{t(n-1)} \mid q^2 = p^{2e}$, therefore we get $t(n-1) \leq 2e$ and $6e = t(n-1) + t \leq 2e + t$ which means that $4e \leq t = \frac{6e}{n}$. Hence we have the contradiction that $n = 1$. \square

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